STABILITY OF A CYLINDRICAL SHELL IN THE BENDING STATE OF STRESS

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The stability of an infinite cylindrical shell subjected to ring loading is investigated. The solution of the problem is given on the basis of linearization of the near bending state of stress with a subsequent application of the Bubnov-Galerkin method. The numerical analysis is carried out on an electronic digital computer. The cases of ring loading acting on an infinite shell (Fig. 1), of ring loading acting on a semi-infinite shell (Fig. 2), and of a system of moments distributed uniformly over the endface (Fig. 3) are considered. In all cases the critical loading and the number of waves at buckling are determined.

1. Let us start from the following relationships for the strain components

$$\varepsilon_1 = \frac{\partial u}{\partial \alpha} + \frac{1}{2}\left(\frac{\partial w}{\partial \alpha}\right)^2, \qquad \varepsilon_2 = \frac{\partial v}{\partial \beta} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial \beta}\right)^2, \qquad \gamma = \frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} + \frac{\partial w}{\partial \alpha}\frac{\partial w}{\partial \beta}$$

$$\varkappa_1 = -\frac{\partial^2 w}{\partial \alpha^2}, \qquad \varkappa_2 = -\frac{\partial^2 w}{\partial \beta^2}, \qquad \tau = -\frac{\partial^2 w}{\partial \alpha\, \partial \beta} \tag{1.1}$$

Here u, v are displacements along the coordinate lines α, β; w along the normal, where w is positive if the displacement is towards the center of curvature; R is the shell radius.

As is known, the strain potential energy of a shell is composed of the strain energy in the middle surface and the bending energy

$$U_1 = \frac{E_1}{2}\int_{\Omega}\left(\varepsilon_1^2 + \varepsilon_2^2 + 2\sigma\varepsilon_1\varepsilon_2 + \frac{1-\sigma}{2}\gamma^2\right) d\alpha\, d\beta \qquad \left(E_1 = \frac{Eh}{1-\sigma^2}\right) \tag{1.2}$$

$$U_2 = \frac{E_2}{2}\int_{\Omega}\left[\varkappa_1^2 + \varkappa_2^2 + 2\sigma\varkappa_1\varkappa_2 - 2(1-\sigma)\tau^2\right] d\alpha\, d\beta \qquad \left(E_2 = \frac{Eh^3}{12(1-\sigma^2)}\right)$$

Here h is the thickness, σ the Poisson coefficient, E the Young's modulus.

In conformity with (1.2), the stress resultants T_1, T_2, S and the moments M_1, M_2, M are connected to the strain components by Hooke's law

$$T_1 = E_1(\varepsilon_1 + \sigma\varepsilon_2), \quad T_2 = E_1(\varepsilon_2 + \sigma\varepsilon_1), \quad S = {}^1/_2\, E_1(1-\sigma)\gamma$$

$$M_1 = E_2(\varkappa_1 + \sigma\varkappa_2), \quad M_2 = E_2(\varkappa_2 + \sigma\varkappa_1), \quad M = E_2(1-\sigma)\tau \tag{1.3}$$

On the basis of the Lagrange variational principle, we obtain the equilibrium equations

$$\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} = 0, \qquad \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} = 0 \tag{1.4}$$

$$\frac{\partial^2 M_1}{\partial \alpha^2} + 2\frac{\partial^2 M}{\partial \alpha\, \partial \beta} + \frac{\partial^2 M_2}{\partial \beta^2} + \frac{T_2}{R} + \frac{\partial^2 w}{\partial \alpha^2} T_1 + 2\frac{\partial^2 w}{\partial \alpha\, \partial \beta} S + \frac{\partial^2 w}{\partial \beta^2} T_2 = 0 \tag{1.5}$$

These equations are valid on portions of the shell where the load is absent.

If we introduce the dimensionless parameters

$$u_1 = \frac{u}{h}, \quad v_1 = \frac{v}{h}, \quad w_1 = \frac{w}{h}, \quad \lambda^2 = \sqrt{3(1-\sigma^2)}\,\frac{R}{h}, \quad \xi = \frac{\alpha}{R}, \quad \eta = \frac{\beta}{R}$$

then by taking account of (1.1), (1.3) we write the system (1.4), (1.5) in displacements

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{1-\sigma}{2}\frac{\partial^2}{\partial \eta^2}\right) u_1 + \frac{1+\sigma}{2}\frac{\partial^2}{\partial \xi\, \partial \eta} v_1 - \sigma \frac{\partial}{\partial \xi} w_1 + \frac{\sqrt{3(1-\sigma^2)}}{\lambda^2}\left(\frac{\partial w_1}{\partial \xi}\frac{\partial^2 w_1}{\partial \xi^2} + \right.$$

$$\left. + \frac{1-\sigma}{2}\frac{\partial w_1}{\partial \xi}\frac{\partial^2 w_1}{\partial \eta^2} + \frac{1+\sigma}{2}\frac{\partial w_1}{\partial \eta}\frac{\partial^2 w_1}{\partial \xi\, \partial \eta}\right) = 0$$

$$\frac{1+\sigma}{2}\frac{\partial^2}{\partial \xi \partial \eta} u_1 + \left(\frac{1-\sigma}{2}\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) v_1 - \frac{\partial}{\partial \eta} w_1 + \frac{\sqrt{3(1-\sigma^2)}}{\lambda^2}\left(\frac{\partial w_1}{\partial \eta}\frac{\partial^2 w_1}{\partial \eta^2} + \right.$$

$$\left. + \frac{1-\sigma}{2}\frac{\partial w_1}{\partial \eta}\frac{\partial^2 w_1}{\partial \xi^2} + \frac{1+\sigma}{2}\frac{\partial w_1}{\partial \xi}\frac{\partial^2 w_1}{\partial \xi\, \partial \eta}\right) = 0 \tag{1.6}$$

$$\sigma \frac{\partial}{\partial \xi} u_1 + \frac{\partial}{\partial \eta} v_1 - \left(1 - \frac{1-\sigma^2}{4\lambda^4}\nabla^4\right) w_1 + \frac{\sqrt{3(1-\sigma^2)}}{\lambda^2}\left[\frac{\sigma}{2}\left(\frac{\partial w_1}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial w_1}{\partial \eta}\right)^2 + \right.$$

$$+ \sigma w_1 \frac{\partial^2 w_1}{\partial \xi^2} + w_1 \frac{\partial^2 w_1}{\partial \eta^2} - \frac{\partial u_1}{\partial \xi}\frac{\partial^2 w_1}{\partial \xi^2} - \sigma \frac{\partial u_1}{\partial \xi}\frac{\partial^2 w_1}{\partial \eta^2} - (1-\sigma)\frac{\partial u_1}{\partial \eta}\frac{\partial^2 w_1}{\partial \xi\, \partial \eta} - \sigma \frac{\partial v_1}{\partial \eta}\frac{\partial^2 w_1}{\partial \xi^2} -$$

$$\left. - \frac{\partial v_1}{\partial \eta}\frac{\partial^2 w_1}{\partial \eta^2} - (1-\sigma)\frac{\partial v_1}{\partial \xi}\frac{\partial^2 w_1}{\partial \xi \partial \eta}\right] - \frac{3(1-\sigma^2)}{\lambda^4}\left[\frac{1}{2}\left(\frac{\partial w_1}{\partial \xi}\right)^2 \frac{\partial^2 w_1}{\partial \xi^2} + \frac{\sigma}{2}\left(\frac{\partial w_1}{\partial \eta}\right)^2 \frac{\partial^2 w_1}{\partial \xi^2} + \right.$$

$$\left. + \frac{1}{2}\left(\frac{\partial w_1}{\partial \eta}\right)^2 \frac{\partial^2 w_1}{\partial \eta^2} + \frac{\sigma}{2}\left(\frac{\partial w_1}{\partial \xi}\right)^2 \frac{\partial^2 w_1}{\partial \eta^2} + (1-\sigma)\frac{\partial w_1}{\partial \xi}\frac{\partial w_1}{\partial \eta}\frac{\partial^2 w_1}{\partial \xi\, \partial \eta}\right] = 0$$

$$\nabla^4 = \frac{\partial^4}{\partial \xi^4} + 2\frac{\partial^4}{\partial \xi^2\, \partial \eta^2} + \frac{\partial^4}{\partial \eta^4}$$

Expressing T_1, T_2, S in terms of the stress function Φ by means of the known formulas

$$T_1 = \frac{\partial^2 \Phi}{\partial \beta^2}, \qquad T_2 = \frac{\partial^2 \Phi}{\partial \alpha^2}, \qquad S = -\frac{\partial^2 \Phi}{\partial \alpha\, \partial \beta} \tag{1.7}$$

we arrive at a system of equations in the deflection and stress function

$$\frac{1}{12(1-\sigma^2)}\nabla^4 w_1 - \frac{\partial^2 w_1}{\partial \xi^2}\frac{\partial^2 \varphi}{\partial \eta^2} + 2\frac{\partial^2 w_1}{\partial \xi\, \partial \eta}\frac{\partial^2 \varphi}{\partial \xi \partial \eta} - \frac{\partial^2 w_1}{\partial \eta^2}\frac{\partial^2 \varphi}{\partial \xi^2} - \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}\frac{\partial^2 \varphi}{\partial \xi^2} = 0$$

$$\nabla^4 \varphi - \left(\frac{\partial^2 w_1}{\partial \xi\, \partial \eta}\right)^2 + \frac{\partial^2 w_1}{\partial \xi^2}\frac{\partial^2 w_1}{\partial \eta^2} + \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}\frac{\partial^2 w_1}{\partial \xi^2} = 0 \qquad \left(\varphi = \frac{\Phi}{Eh^3}\right) \tag{1.8}$$

The first equation of the system (1.8) is obtained from (1.5) taking account of (1.7), and the second expresses the condition of comatibility of the strains.

2. In the axisymmetric case, (1.6) and (1.8) become, respectively,

$$\frac{d^2 u_0}{d\xi^2} - \sigma \frac{dw_0}{d\xi} + \frac{\sqrt{3(1-\sigma^2)}}{\lambda^2}\frac{dw_0}{d\xi}\frac{d^2 w_0}{d\xi^2} = 0 \tag{2.1}$$

$$\sigma \frac{du_0}{d\xi} - \left(1 - \frac{1-\sigma^2}{4\lambda^4}\frac{d^4}{d\xi^4}\right) w_0 + \frac{\sqrt{3(1-\sigma^2)}}{\lambda^2}\left[\frac{\sigma}{2}\left(\frac{dw_0}{d\xi}\right)^2 - \sigma w_0 \frac{d^2 w_0}{d\xi^2} - \right.$$

$$\left. - \frac{du_0}{d\xi}\frac{d^2 w_0}{d\xi^2}\right] - \frac{3(1-\sigma^2)}{2\lambda^4}\left(\frac{dw_0}{d\xi}\right)^2 \frac{d^2 w_0}{d\xi^2} = 0 \tag{2.2}$$

$$\frac{d^4 \varphi_0}{d\xi^4} + \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}\frac{d^2 w_0}{d\xi^2} = 0 \tag{2.3}$$

$$\frac{1}{12(1-\sigma^2)}\frac{d^4w_0}{d\xi^4}-\frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}\frac{d^2\varphi_0}{d\xi^2}=0 \tag{2.4}$$

Eliminating u_0 from (2.1),(2.2), φ_0 from (2.3),(2.4), we obtain Eq.

$$\frac{d^4w_0}{d\xi^4}+4\lambda^4w_0=0 \tag{2.5}$$

As is known, the general solution of (2.5) is

$$w_0=e^{\lambda\xi}(C_1\cos\lambda\xi+C_2\sin\lambda\xi)+e^{-\lambda\xi}(C_3\cos\lambda\xi+C_4\sin\lambda\xi) \tag{2.6}$$

Since the forces applied at $\xi=0$ produce a local strain which vanishes rapidly as the distance ξ increases, the first term on the right side of (2.6) sould vanish. Hence, $C_1=C_2=0$ and w_0 is finally written as

$$w_0=b\chi+\bar{b}\bar{\chi}\ (\chi=e^{-\bar{\mu}\lambda\xi},\ \bar{\chi}=e^{\bar{\mu}\lambda\xi},\ \mu=1+i,\ \bar{\mu}=1-i)$$

The constants b and $\bar{b}$ are determined from the boundary conditions at $\xi=0$. From (2.1),(2.4) we find

$$\frac{du_0}{d\xi}=\sigma(b\chi+\bar{b}\bar{\chi})-\frac{\sqrt{3(1-\sigma^2)}}{2\lambda^2}(b\mu\chi+\bar{b}\bar{\mu}\bar{\chi})^2 \tag{2.8}$$

$$\frac{d^2\varphi_0}{d\xi^2}=-\frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}(b\chi+\bar{b}\bar{\chi}) \tag{2.9}$$

The exact solution of the axisymmetric problem has therefore been obtained.

3. We seek the general solution of the system (1.6) as

$$u_1=u_0+u^*,\qquad v_1=v_0+v^*,\quad w_1=w_0+w^* \tag{3.1}$$

and we write the general solution of (1.8) as

$$w_1=w_0+w^*,\quad \varphi=\varphi_0+\varphi^* \tag{3.2}$$

Here, u_0, v_0, w_0, φ_0 correspond to the axisymmetric case.

After substituting (3.1) and (3.2) into the system (1.6) and (1.8) having taken account of (2.7),(2.8),(2.9), and linearized the obtained equations near the axisymmetric bending state, we arrive at differential equations with variable coefficents of u^*, v^*, w^*

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial\xi^2}-\frac{1-\sigma}{2}\frac{\partial^2}{\partial\eta^2}\right)u^*+\frac{1+\sigma}{2}\frac{\partial^2}{\partial\xi\,\partial\eta}v^*-\sigma\frac{\partial}{\partial\xi}w^*+\\
&+\frac{\sqrt{3(1-\sigma^2)}}{\lambda}\left[\lambda(b\mu^2\chi+\bar{b}\bar{\mu}^2\bar{\chi})\frac{\partial w^*}{\partial\xi}-(b\mu\chi+\bar{b}\bar{\mu}\bar{\chi})\left(\frac{\partial^2w^*}{\partial\xi^2}+\frac{1-\sigma}{2}\frac{\partial^2w^*}{\partial\eta^2}\right)\right]=0\\
&\frac{1+\sigma}{2}\frac{\partial^2}{\partial\xi\,\partial\eta}u^*+\left(\frac{1-\sigma}{2}\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)v^*-\frac{\partial}{\partial\eta}w^*+\\
&+\frac{\sqrt{3(1-\sigma^2)}}{2\lambda}\left[(1-\sigma)\lambda(b\mu^2\chi+\bar{b}\bar{\mu}^2\bar{\chi})\frac{\partial w^*}{\partial\eta}-(1+\sigma)(b\mu\chi+\bar{b}\bar{\mu}\bar{\chi})\frac{\partial^2w^*}{\partial\xi\,\partial\eta}\right]=0\\
&\sigma\frac{\partial}{\partial\xi}u^*+\frac{\partial}{\partial\eta}v^*-\left(1-\frac{1-\sigma^2}{4\lambda^4}\nabla^4\right)w^*+\\
&+\frac{\sqrt{3(1-\sigma^2)}}{\lambda^2}\left[\lambda^2(b\mu^2\chi+\bar{b}\bar{\mu}^2\bar{\chi})\left(\sigma w^*-\frac{\partial u^*}{\partial\xi}-\sigma\frac{\partial v^*}{\partial\eta}\right)-\right.\\
&-\sigma\lambda(b\mu\chi+\bar{b}\bar{\mu}\bar{\chi})\frac{\partial w^*}{\partial\xi}+(b\chi+\bar{b}\bar{\chi})(1-\sigma^2)\frac{\partial^2w^*}{\partial\eta^2}+\\
&\left.+\frac{3(1-\sigma^2)}{2\lambda}(b\mu^2\chi+\bar{b}\bar{\mu}^2\bar{\chi})(b\mu\chi+\bar{b}\bar{\mu}\bar{\chi})\frac{\partial w^*}{\partial\xi}\right]=0
\end{aligned} \tag{3.3}$$

and a system in w^*, φ^*

$$\nabla^4\varphi^* + \lambda^2(b\mu^2\chi + \overline{b\mu^2\chi})\frac{\partial^2 w^*}{\partial\eta^2} + \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}\frac{\partial^2 w^*}{\partial\xi^2} = 0$$

$$\frac{1}{12(1-\sigma^2)}\nabla^4 w^* - \lambda^2(b\mu^2\chi + \overline{b\mu^2\chi})\frac{\partial^2\varphi^*}{\partial\eta^2} + \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}(b\chi + \overline{b\chi})\frac{\partial^2 w^*}{\partial\eta^2} - \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}\frac{\partial^2\varphi^*}{\partial\xi^2} = 0 \qquad (3.4)$$

Thus, the problem has been reduced to the solution of either the system (3.3) or the system (3.4) with appropriate boundary conditions.

We apply the Bubnov-Galerkin method to solve the systems of Eqs. (3.3) and (3.4). We take u^*, v^*, w^*, φ^* as $u^* = f_1(\xi)\cos n\eta$, $v^* = f_2(\xi)\sin n\eta$, $w^* = f_3(\xi)\cos n\eta$, $\varphi^* = f_4(\xi)\cos n\eta$, where n is the number of complete waves around the circumference. Then (3.3) becomes

$$f_1'' - \frac{1-\sigma}{2}n^2 f_1 + \frac{1+\sigma}{2}nf_2' - \sigma f_3' + \frac{\sqrt{3(1-\sigma^2)}}{\lambda}\Big[\lambda(b\mu^2\chi + \overline{b\mu^2\chi})f_3' - (b\mu\chi + \overline{b\mu\chi})\Big(f_3'' - \frac{1-\sigma}{2}n^2 f_3\Big)\Big] = 0 \qquad (3.5)$$

$$\frac{1+\sigma}{2}nf_1' + n^2 f_2 - \frac{1-\sigma}{2}f_2'' - nf_3 + \frac{\sqrt{3(1-\sigma^2)}}{2\lambda}\big[(1-\sigma)\lambda(b\mu^2\chi + \overline{b\mu^2\chi})nf_3 - (1+\sigma)(b\mu\chi + \overline{b\mu\chi})nf_3'\big] = 0 \qquad (3.6)$$

$$\sigma f_1' + nf_2 - f_3 + \frac{1-\sigma^2}{4\lambda^4}(f_3^{IV} - 2n^2 f_3'' + n^4 f_3) + \frac{\sqrt{3(1-\sigma^2)}}{\lambda^2}\big[\lambda^2(b\mu^2\chi + \overline{b\mu^2\chi}) \times (\sigma f_3 - f_1' - \sigma n f_2) - \sigma\lambda(b\mu\chi + \overline{b\mu\chi})f_3' - (1-\sigma^2)(b\chi + \overline{b\chi})n^2 f_3\big] + \frac{3(1-\sigma^2)}{2\lambda}(b\mu^2\chi + \overline{b\mu^2\chi})(b\mu\chi + \overline{b\mu\chi})f_3' = 0 \qquad (3.7)$$

and Eqs. (3.4) go over into the following:

$$f_4^{IV} - 2n^2 f_4'' + n^4 f_4 - \lambda^2(b\mu^2\chi + \overline{b\mu^2\chi})n^2 f_4 + \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}f_3'' = 0 \qquad (3.8)$$

$$\frac{1}{12(1-\sigma^2)}(f_3^{IV} - 2n^2 f''_3 + n^4 f_3) + \lambda^2(b\mu^2\chi + \overline{b\mu^2\chi})n^2 f_4 - \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}(b\chi + \overline{b\chi})n^2 f_3 - \frac{\lambda^2}{\sqrt{3(1-\sigma^2)}}f_4'' = 0 \qquad (3.9)$$

The functions $f_1(\xi)$, $f_2(\xi)$, $f_3(\xi)$, $f_4(\xi)$ should satisfy the assigned boundary conditions. Let us select the following approximate expression for $f_3(\xi)$

$$f_3(\xi) = a_1\chi + \bar{a}_1\bar{\chi} + a_2\chi^2 + \bar{a}_2\bar{\chi}^2 + a_3\chi\bar{\chi}$$

where the a_i are some constants.

Hence, a_1, $\bar{a}_1$ can be expressed from the boundary conditions in terms of a_2, $\bar{a}_2$, a_3 as

$$a_1 = \gamma_1 a_2 + \gamma_2\bar{a}_2 + \gamma_3 a_3, \quad \bar{a}_1 = \bar{\gamma}_2 a_2 + \bar{\gamma}_1\bar{a}_2 + \bar{\gamma}_3 a_3$$

Therefore, $f_3(\xi)$ depends on the three parameters a_2, $\bar{a}_2$, a_3

$$f_3(\xi) = (\bar{\gamma}_1\bar{\chi} + \gamma_2\chi + \chi^2)a_2 + (\bar{\gamma}_2\bar{\chi} + \bar{\gamma}_1\chi + \chi^2)\ \bar{a}_2 + (\gamma_3\chi + \bar{\gamma}_3\bar{\chi} + \chi\bar{\chi})a_3 \qquad (3.10)$$

After substituting (3.10) into (3.5), (3.6) and (3.8), we obtain a system to determine $f_1(\xi)$, $f_2(\xi)$

$$f_1'' - {}^1/_2(1-\sigma)n^2 f_1 + {}^1/_2(1+\sigma)nf_2' = A_1\chi + \bar{A}_1\bar{\chi} + B_1\chi^2 + \bar{B}_1\bar{\chi}^2 + K_1\chi\bar{\chi} + L_1\chi^3 + \bar{L}_1\bar{\chi}^3 + N_1\chi^2\bar{\chi} + \bar{N}_1\bar{\chi}^2\chi$$

$$\tfrac{1}{2}(1+\sigma)nf_1' - \tfrac{1}{2}(1-\sigma)f_2'' + n^2 f_2 = A_2\chi + \bar{A}_2\bar{\chi} + B_2\chi^2 + \bar{B}_2\bar{\chi}^2 + K_2\chi\bar{\chi} + L_2\chi^3 + \bar{L}_2\bar{\chi}^3 + N_2\chi^2\bar{\chi} + \bar{N}_2\bar{\chi}^2\chi \tag{3.11}$$

and an equation to determine $f_4(\xi)$ (3.12)

$$f_4^{IV} - 2n^2 f_4'' + n^4 f_4 = A_3\chi + \bar{A}_3\bar{\chi} + B_3\chi^2 + \bar{B}_3\bar{\chi}^2 + K_3\chi\bar{\chi} + L_3\chi^3 + \bar{L}_3\bar{\chi}^3 + N_3\chi^2\bar{\chi} + \bar{N}_3\bar{\chi}^2\chi$$

Let us find the general solution of the system (3.11) and (3.12)

$$f_1(\xi) = \tfrac{1}{2}(1+\sigma)n[nC_1 + (\xi n - 1)C_2]e^{-n\xi} + A_1^*\chi + \bar{A}_1^*\bar{\chi} + B_1^*\chi^2 + \bar{B}_1^*\bar{\chi}^2 + K_1^*\chi\bar{\chi} + L_1^*\chi^3 + \bar{L}_1^*\bar{\chi}^3 + N_1^*\chi^2\bar{\chi} + \bar{N}_1^*\bar{\chi}^2\chi \tag{3.13}$$

$$f_2(\xi) = \tfrac{1}{2}n\{(1+\sigma)nC_3 + [(1+\sigma)n\xi - 4]C_4\}e^{-n\xi} + A_2^*\chi + \bar{A}_2^*\bar{\chi} + B_2^*\chi^2 + \bar{B}_2^*\bar{\chi}^2 + K_2^*\chi\bar{\chi} + L_2^*\chi^3 + \bar{L}_2^*\bar{\chi}^3 + N_2^*\chi^2\bar{\chi} + \bar{N}_2^*\bar{\chi}^2\chi \tag{3.14}$$

$$f_4(\xi) = (C_5 + C_6\xi)e^{-n\xi} + A_3^*\chi + \bar{A}_3^*\bar{\chi} + B_3^*\chi^2 + \bar{B}_3^*\bar{\chi}^2 + K_3^*\chi\bar{\chi} + L_3^*\chi^3 + \bar{L}_3^*\bar{\chi}^3 + N_3^*\chi^2\bar{\chi} + \bar{N}_3^*\bar{\chi}^2\chi \tag{3.15}$$

The constants C_i are determined from the boundary conditions; the polinomials A_i^*, B_i^*, K_i^*, L_i^*, N_i^* $(i = 1, 2, 3)$ depend on q, n, λ, σ, a_2, $\bar{a}_2$, a_3; the bar denotes the complex conjugate.

Substituting (3.10),(3.13),(3.14) into (3.7) or (3.10),(3.15) into (3.9), alternately multiplying the equation obtained by $(\gamma_1\chi + \overline{\gamma_2\chi} + \chi_2)d\xi$, $(\gamma_2\chi + \overline{\gamma_1\chi} + \bar{\chi}^2)\,d\xi$, $(\gamma_3\chi + \overline{\gamma_3\chi} + \chi\bar{\chi})d\xi$ and integrating between zero and infinity we obtain three equations in a_2, $\bar{a}_2$, a_3 in each case

$$\begin{aligned} S_1(q, n, \lambda, \sigma)a_2 + S_2(q, n, \lambda, \sigma)\bar{a}_2 + S_3(q, n, \lambda, \sigma)\,a_3 &= 0 \\ \bar{S}_2(q, n, \lambda, \sigma)\,a_2 + \bar{S}_1(q, n, \lambda, \sigma)\,\bar{a}_2 + \bar{S}_3(q, n, \lambda, \sigma)a_3 &= 0 \\ S_4(q, n, \lambda, \sigma)a_2 + \bar{S}_4(q, n, \lambda, \sigma)\bar{a}_2 + S_5(q, n, \lambda, \sigma)\,a_3 &= 0 \end{aligned} \tag{3.16}$$

where the S_i $(i = 1,2,3,4,5)$ are second order polynomials in q, the loading parameter. Since $a_2 \neq \bar{a}_2 \neq a_3 \neq 0$, then to obtain a nontrivial solution of the system (3.16) its determinant should be zero

$$\begin{vmatrix} S_1 & S_2 & S_3 \\ \bar{S}_2 & \bar{S}_1 & \bar{S}_3 \\ S_4 & \bar{S}_4 & S_5 \end{vmatrix} = 0$$

For the given σ we arrive at a sixth order algebraic equation in q which depends on $k = n/\lambda$,

$$F_1(k)q^6 + F_2(k)q^5 + F_3(k)q^4 + F_4(k)q^3 + F_5(k)q^2 + F_6(k)q + F_7(k) = 0 \tag{3.17}$$

Eq.(3.17) has been solved numerically by using the "Minsk-12" digital computer for $\sigma = 0.3$. Programs were compiled for the calculation of the coefficients and roots of the equation. For each given λ there was found an n, for which q became a minimum, denoted later by q_0.

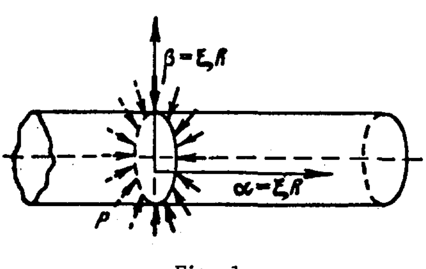

Fig. 1

4. Let us consider an infinite cylindrical shell subjected to uniform (axisymmetric) ring loading (Fig. 1). To solve the stability problem, let us use the equilibrium equations in displacements.

In the axisymmetric case w_0 satisfies the following boundary conditions for $\xi = 0$:

$$\frac{dw_0}{d\xi} = 0, \qquad \frac{d^3 w_0}{d\xi^3} = q \qquad \left(q = \frac{6P(1-\sigma^2)R^3}{Eh^4}\right) \tag{4.1}$$

Then taking account of (4.1), Formulas (2.7), (2.8) become

$$w_0 = \frac{q}{8\lambda^3}(\mu\chi + \bar{\mu}\bar{\chi}), \qquad \frac{du_0}{d\xi} = \frac{\sigma q}{8\lambda^3}(\mu\chi + \bar{\mu}\bar{\chi}) - \frac{q^2\sqrt{3(1-\sigma^2)}}{128\lambda^6}(\mu^2\chi + \bar{\mu}^2\bar{\chi})^2 \tag{4.2}$$

The general solution of the problem is taken in the form (3.1), hence, u^*, v^*, w^* satisfy conditions at $\xi = 0$

$$Q = \frac{\partial w^*}{\partial \xi} = u^* = \frac{\partial v^*}{\partial \xi} = 0$$

where Q is the transverse force. Hence

$$f_1 = f_2' = 0, \quad f_3' = f'''_3 = 0 \qquad \text{for} \quad \xi = 0 \tag{4.3}$$

We take the functions $f_1(\xi)$, $f_2(\xi)$, $f_3(\xi)$ in the form (3.10), (3.13), (3.14). Taking (4.3) into account, we find

$$\gamma_1 = -5, \quad \gamma_2 = 3i, \quad \gamma_3 = {}^1/_2(1 + 3i) \tag{4.4}$$

$$C_1 = {}^1/_4[4 + (1+\sigma)\eta\xi]H, \quad C_2 = {}^1/_4\ \xi\ (1+\sigma)T$$

$$C_3 = {}^1/_4(1+\sigma)\ (1 + n\xi)H, \quad C_4 = {}^1/_4[(3-\sigma) + (1-\sigma)\eta\xi]T/n$$

$$H = -A_1^* - \bar{A}_1^* - B_1^* - \bar{B}_1^* - K_1^* - L_1^* - \bar{L}_1^* - N_1^* - \bar{N}_1^*$$

$$T = \lambda\,[\mu A_2^* + \bar{\mu}\bar{A}_2^* + 2\mu B_2^* + 2\bar{\mu}\bar{B}_2^* + (\mu + \bar{\mu})K_2^* + 3\mu L_2^* + 3\bar{\mu}\bar{L}_2^* + (2\mu + \bar{\mu})N_2^* + (2\bar{\mu} + \mu)\bar{N}_2^*]$$

The coefficients of (3.17) are calculated on the basis of these relations, but are not presented because of their awkwardness.

Let us present the results of calculating q_0 for different values of R/h:

$R/h =$	10	20	50	100	200	400	600	800	1000	5000	10 000
$n =$	3	4	5	7	9	13	16	19	22	43	61
$q_0 =$	0.76	0.53	0.42	0.38	0.38	0.38	0.38	0.38	0.38	0.38.	0.38

The critical loading is hence determined by means of Formula

$$P_* = q_0 \frac{Eh}{(1-\sigma^2)^{0.75}}\left(\frac{h}{R}\right)^{1.5} \tag{4.5}$$

5. Let us consider a semi-infinite cylindrical shell subjected to uniform (axisymmetric) ring loading applied at the endface (Fig. 2).

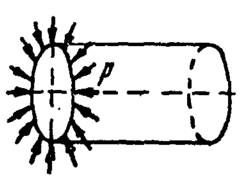

Fig. 2

To solve this problem it is more convenient to use the equilibrium equations in the deflection and the stress function.

In the axisymmetric case w_0 satisfies boundary conditions at $\xi = 0$

$$\frac{d^2 w_0}{d\xi^2} = 0, \qquad \frac{d^3 w_0}{d\xi^3} = q \qquad \left(q = \frac{12P(1-\sigma^2)R^3}{Eh^4}\right) \tag{5.1}$$

Taking account of (5.1), Formulas (2.7), (2.9) become

$$w_0 = \frac{q}{4\lambda^3}(\chi + \bar{\chi}), \qquad \frac{d^2\varphi_0}{d\xi^2} = -\frac{q}{4\lambda\sqrt{3(1-\sigma^2)}}(\chi + \bar{\chi}) \tag{5.2}$$

We take the general solution of the problem in the form (3.2); hence the following conditions should be satisfied at $\xi = 0$:

$$T_1 = S = Q = M = 0 \tag{5.3}$$

Hence, at $\xi = 0$ we have

$$f_4 = f_4^1 = 0$$

$$f_3''' - (2-\sigma) n^2 f_3' = 0, \qquad f_3'' - \sigma n^2 f_3 = 0 \tag{5.4}$$

We take the functions $f_3(\xi)$ and $f_4(\xi)$ in the form (3.10) and (3.15). Taking account of (5.3) and (5.4), we find

$$\gamma_1 = \frac{\lambda}{\Delta}\{2\mu[4\mu^2\lambda^2 - (2-\sigma)n^2](\bar{\mu}^2\lambda^2 - \sigma n^2) - \bar{\mu}[\bar{\mu}^2\lambda^2 - (2-\sigma)n^2](4\mu^2\lambda^2 - \sigma n^2)\}$$

$$\gamma_2 = \frac{\lambda}{\Delta}\{2\bar{\mu}[4\bar{\mu}^2\lambda^2 - (2-\sigma)n^2](\bar{\mu}^2\lambda^2 - \sigma n^2) - \bar{\mu}[\bar{\mu}^2\lambda^2 - (2-\sigma)n^2](4\bar{\mu}^2\lambda^2 - \sigma n^2)\}$$

$$\gamma_3 = \frac{\lambda}{\Delta}\{(\mu+\bar{\mu})[(\mu+\bar{\mu})^2\lambda^2 - (2-\sigma)n^2](\bar{\mu}^2\lambda^2 - \sigma n^2) - \bar{\mu}[\bar{\mu}^2\lambda^2 - (2-\sigma)n^2] \times$$
$$\times [(\mu+\bar{\mu})^2\lambda^2 - \sigma n^2]\}$$

$$\Delta = \lambda\{\bar{\mu}[\bar{\mu}^2\lambda^2 - (2-\sigma)n^2](\mu^2\lambda^2 - \sigma n^2) - \mu[\mu^2\lambda^2 - (2-\sigma)n^2](\bar{\mu}^2\lambda^2 - \sigma n^2)\}$$

$$C_5 = -A_3^* - A_3^* - B_3^* - \bar{B}_3^* - K_3^* - L_3^* - \bar{L}_3^* - N_3^* - \bar{N}_3^*$$

$$C_6 = (\mu\lambda - n)\bar{A}_3^* + (\bar{\mu}\lambda - n)\bar{A}_3^* + (2\mu\lambda - n)B_3^* + (2\bar{\mu}\lambda - n)\bar{B}_3^* + [(\mu+\bar{\mu})\lambda - n]K_3^* +$$
$$+ (3\mu\lambda - n)L_3^* + (3\mu\lambda - n)L_3^* + [(2\mu+\bar{\mu})\lambda - n]N_3^* + [(2\bar{\mu}+\mu)\lambda - n]\bar{N}_3^*$$

Here

$$A_3^* = -\frac{\mu^2\lambda^2}{\sqrt{3(1-\sigma^2)}\,(\mu^2\lambda^2 - n^2)^2}\,a_1, \qquad B_3^* = \frac{qn^2\mu^2}{4\lambda(4\mu^2\lambda^2 - n^2)^2}\,a_1 -$$
$$-\frac{4\mu^2\lambda^2}{\sqrt{3(1-\sigma^2)}\,(4\mu^2\lambda^2 - n^2)^2}\cdot a_2$$

$$K_3^* = \frac{qn^2}{4\lambda[(\mu+\bar{\mu})^2\lambda^2 - n^2]^2}(\bar{\mu}^2 a_1 + \mu^2\bar{a}_1) - \frac{(\mu+\bar{\mu})^2\lambda^2}{\sqrt{3(1-\sigma^2)}\,[(\mu+\bar{\mu})^2\lambda^2 - n^2]^2}\,a_3$$

$$L_3^* = \frac{qn^2\mu^2}{4\lambda(9\mu^2\lambda^2 - n^2)^2}\,a_2, \qquad N_3^* = \frac{qn^2}{4\lambda[(2\mu+\bar{\mu})^2\lambda^2 - n^2]^2}\cdot(\bar{\mu}^2 a_2 + \mu^2\bar{a}_2)$$

On the basis of these relationships, we calculated the coefficients of (3.17) but do not present them because of their awkwardness.

Let us present the results of calculating q_0 for a number of R/h values

$R/h =$	10	20	50	100	200	400	600	800	1000	5000	10 000
$n =$	3	4	5	7	10	13	15	19	21	44	61
$q_0 =$	0.35	0.27	0.22	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18

The critical loading is detrmined from Formula

$$P_* = q_0 \frac{Eh}{(1-\sigma^2)^{0.5}}\left(\frac{h}{R}\right)^{1.5} \tag{5.5}$$

6. Let us consider a semi-infinite cylindrical shell subjected to a system of moments distributed uniformly over the endface (Fig. 3).

The problem is solved analogously to the preceding by using the equilibrium equations in the deflection and stress functions.

Let us present the results of calculating q_0 for a number of R/h values:

$R/h =$	10	20	50	100	200	400	600	800	1000	5000	10 000
$n =$	2	4	5	6	9	13	16	19	22	43	62
$q_0 =$	0.42	0.33	0.29	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21

where

$$M_* = \frac{Eh^2}{(1-\sigma^2)^{0.75}}\,\frac{h}{R} \tag{6.1}$$

To determine n for $R/h \geqslant 100$ we can use Formula

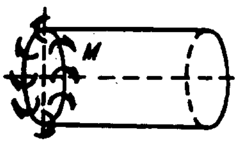

Fig. 3

$$n = \left(\frac{R}{eh}\right)^{1/2}, \qquad e = 2.718$$

For sufficiently large R / h the quantity q_0 is dependent on the thickness, and in the case of the effect of a ring loading on an infinite or semi-infinite shell will equal, respectively, $q_0 = 0.38$ and $q_0 = 0.18$, according to calculations utilizing (4.5) and (5.5).

When a system of moments distributed uniformly over the endface acts, the calculations yield $q_0 = 0.21$ according to (6.1).

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ENERGY CRITERION OF THE STABILITY OF ELASTIC BODIES WHICH DOES NOT REQUIRE THE DETERMINATION OF THE INITIAL STRESS-STRAIN STATE

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It is shown that if the initial state of stress of a body is described by linear elasticity theory, then an energy criterion for neutral equilibrium can be formulated directly in terms of the external loading and the governing bifurcation of the displacements. To do this, besides the fundamental first order displacements, additional second order displacements on which external potential forces perform work during buckling, are introduced to describe the deflected equilibrium position of the body. These additional quadratic displacements are expressed in terms of the first order displacements. It therefore turns out that the stability problem of an elastic body can be solved without a preliminary determination of its initial state of stress. The result obtained can be considered as the foundation and extension of the energy stability criterion in the form of S. P. Timoshenko.

The energy stability criterion which does not require the initial stress determination